GG Summary + Formulas

1. Texture Mapping

$M \to (u,v) \to (R,G,B)$ (x,y,z)

Harmonic: 1: map boundary to polygon 2. min energy $E_y = \frac{1}{2} k_{ij} \|v_i - v_j\|$

- uniform 1
- cord $\frac{1}{\|v_i - v_j\|}$
- cotan $\cot\phi_{ij} + \cot\phi_{ji}$

2. Ray tracing

Homogeneous: $w=0$ vector, $=1$ point, ? normal

Assumptions: no diffr/polariz/interf, straight vacuum

Fw/Bw raytracing

ray: $r(t) = o + td$

sphere: $\|x-c\|^2 - r^2 = 0$

plane: $ax+by+cz+d=0$, $(x-p).n = 0$

triangle: $x = s_1 p_1 + s_2 p_2 + s_3 p_3$

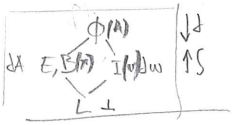
Circle $A = 2\pi r$, $V = \pi r^2$ $\int \frac{d}{dr}$

Sphere $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$ $\int \frac{d}{dr}$

3. Light & Matter

Blackbody Plank

Light emission
- incandescence (heat)
- luminescence (chem/elektr)
- flu/phosphorescence
- chemi/bioluminescence



Radiometry

flux	$\Phi(A)$	$\frac{J}{s} = W$	$\Phi(A) = \int_A E(x)\,dA$	
Irradiance	$E(x) = \frac{\partial\Phi(A)}{\partial A(x)}$	$\frac{W}{m^2}$	$E(x) = \int_{H^2} L(x,\omega)\cos\theta\,d\omega$	
Radiosity	$B(x) =$ "			
Rad. Intens	$I(\omega) = \frac{\partial\Phi}{\partial\omega}$	$\frac{W}{sr}$	$\Phi = \int_{S^2} I(\omega)\,d\omega$	
Radiance	$L(x,\omega) = \frac{\partial I(\omega)}{\partial A^{\perp}(x,\omega)}$	$\frac{W}{m^2 sr}$	$= \frac{\partial^2\Phi(A)}{\partial\omega\,\partial A(x)}\cos\theta$	

BRDF $f_r(x,\omega_i,\omega_o) = \frac{dL_r(x,\omega_o)}{dE_i(x,\omega_i)} = \frac{dL_r(x,\omega_o)}{L(x,\omega_i)\cos\theta_i\,d\omega_i}$ $+L_e(x,\omega_r)$

$$L_r(x,\omega_r) = \int_{H^2} f_r(x,\omega_r,\omega_i)\,L(x,\omega_i)\cos\theta_i\,d\omega_i$$

isotropic / anisotropic → not

Lambert $L_r(x,\omega_r) = f_r\,E(x)$ | $E(x) = B(x) \Leftrightarrow$ all is reflected $= L_r(x)\,\pi$ | $\int_{H^2}\cos\theta\,d\omega = \pi$

ρ → albedo: % of reflected rad $\frac{\rho}{\pi} = f_r$

Specular Reflection

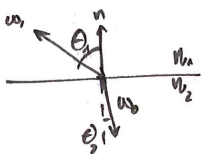
$\theta_i = \theta_o$

Fermat: take least time

Specular Refraction Snell's law

index of refraction $\eta = \frac{v_{vacuum}}{v_{material}}$

$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \Rightarrow \frac{\eta_1}{\eta_2} = \frac{\sin\theta_2}{\sin\theta_1}$



$\omega_r = ..$

$\eta_t(\lambda, \text{polariz})$

$\frac{\eta_1}{\eta_2} \gg 1 \to$ mirror

Fresnel: how much

$$P_{\parallel} = \frac{\eta_2\cos\theta_1 - \eta_1\cos\theta_2}{\eta_2\cos\theta_1 + \eta_1\cos\theta_2}$$

$$P_{\perp} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

refl $F_r = \frac{1}{2}(P_{\parallel}^2 + P_{\perp}^2)$

refract $F_t = 1 - F_r$

$$f_r(x,\omega_i,\omega_r) = F_r(\omega_i)\,\frac{\delta(\omega_r - R(\omega_i,n))}{\cos\theta_i}$$ - delta

4. Appearance Modeling

fresnel shadowing microfacet distrib

microfacet surface $f(\omega_i,\omega_o) = \frac{F(\omega_i,\omega_o)\,G(\omega_i,\omega_o)\,D(\omega_h)}{4\,|(\omega_i\cdot n)(\omega_o\cdot n)|}$

Beckmann $D(\omega_h) = \frac{1}{\pi\alpha^2\cos^4\theta_h}\exp\left(-\tan^2\theta_h/\alpha^2\right)$

5. Monte Carlo

$$F := \int_\Omega f(x)dx = \int_\Omega \frac{f(x)}{p(x)}p(x)\,dx = E_{x\sim p(x)}\left[\frac{f(x)}{p(x)}\right] \overset{\sim}{=} \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

Inversion method
1) Compute CDF $P(x) = \int^x p(x')dx'$
2) invert $P^{-1}(x)$
3) $\xi \sim U(0,1)$
4) $p: x_i = P^{-1}(\xi)$

Transformation

bijective $T: x_i \to y_i$, $x_i \sim p_x(x) \to y_i = T(x_i) \sim p_y(y) = p_y(T(x)) = \frac{p_x(x)}{|\det J_T(x)|}$

J_T = jacobian = matrix of all 1st derivative

$J_T(x) = \frac{dy}{dx}$

2D: $P(x,y)$ marginal $p_x(x) = \int p(x,y)dy$ $p_y(y|x) = p(x,y)/p(x)$

eg sample disk $r_i = \sqrt{\xi_1}$, $\theta_i = 2\pi\xi_2$

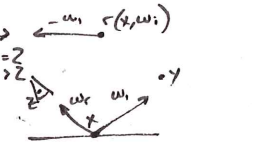
Ambient Occlusion $L_r(x) = \frac{\rho}{\pi}\int_{H^2} V(x,\omega_i)\cos\theta_i\,d\omega_i$

Importance Sample $p(x) = c\,f(x) \to 1 = \int p = c\int f \to \int f = \frac{1}{c}$
↳ reduce variance
average visibility $\to AO = \frac{\rho}{N}\sum_{i=1}^{N} V(x,\omega_i)$ $p = \frac{\cos\theta}{\pi}$

Random sampling • stratified (< var) • latin hypercube (shuffle rows/cols) • multi jittered

6. DIRECT ILLUMINATION

DI: $L_i(x,\omega_i) = L_e(r(x,\omega_i),-\omega_i)$ Ind.!



Light path: light tracing, path tracing, bid. path tr

Integrate: H^2 ⌒ A ▭ $L_r(x,z) = \int_A f_r(x,y,z)L_e(y,x)G(x,y)dA$

$d\omega = \frac{dA\cos\theta'}{\|x-y\|^2}$, $G(x,y) = V(x,y)\frac{|\cos\theta_i||\cos\theta_o|}{\|x-y\|^2}$ geom visib

change of variable transf.

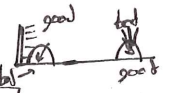
$V = \begin{cases}1 & \text{visible}\\ 0 & \text{else}\end{cases}$

Light src
- point $(p,\phi) \to L_e(p,\omega) = \frac{\Phi}{4\pi}$ } delta
- spot! $(p, I(\omega))$
- direction ω, $L(\omega)$
- sphere (p_{11}, ϕ)
- mesh $p_A(x) = \frac{A(x)}{\sum A(x)}$

$\to p_A(y) = \frac{\cos\theta_o}{\|y-x\|^2}\,p_{S^2}(\omega_i)$

multiple strategies

cosine!: $p(\omega) = \frac{\cos\theta}{\pi}$
uniform: $\frac{1}{2\pi}$



var: additive

$$F^N = \sum_{i=1}^{n_s}\frac{1}{n_s}\sum_{j=1}^{n_i}\omega_i(x_{ij})\frac{f(x_{ij})}{p(x_{ij})}$$

$\omega_i(x) = \frac{n_i p_i(x)}{\sum_i n_i p_i(x)}$

n_s = #sampling strat

Environment emit
• lat-long / cubemap / ..

$L_e(x,\omega_i) = L_{env}(\omega_i).V(x,\omega_i)$
$p(\theta,\phi) \propto L_{env}(\theta,\phi)\sin\theta\,d\theta\,d\phi$

1) scalar $L' = L.\sin\theta$ (grayscale)
2) marginal & condit $p(\theta)$, $p(\phi|\theta)$

Advanced Cam • pinhole / thinlens

7. Global Illumination

Light paths: EDSL | Direct L(S|D)E | Indir L(S|D)(S|D)*E | All L(S|D)* E | Caustics LS+DE

Heckbert + * ? (1) ∨

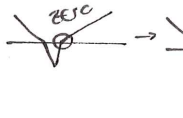
Recursive RCRT $L(x,\omega) = L_e(x,\omega) + \int_A L_e(y,x)\cdot dA + \int_{H^2} .. L_r(r(x,\omega_i),-\omega_i)..d\omega$
direct, emit accept — indirect, recursion

Russian Roulette → termination ← Path Tracing

$q \in [0,1]$, $\xi \sim U(0,1)$ $\xi > q \to$ continue, $\xi \le q \to$ terminate $L_i \to \frac{L_i}{1-q}$

Fails caustics, delta BRDF

Bid. PT ∿LSDSE



8. Many lights rendering

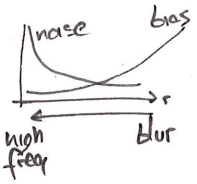
VPL 1) distribute VPL along paths (vertices) 2) approx Ind.L using VPL → unbiased
Singularity → splatches $G(x,q) = V(x,q)\frac{\cos\theta_x \cos\theta_q}{\|x-q\|^2}$ → $\min(b, G(x,q))$ → bias
VSL

9. Photon Mapping

→ biased
Illumination map → store photons
Alg 1. trace photons, store photon map 2. trace cam rays, estimate direct + indirect (photons)
(x_p, ω_p, Φ_p) emission (l, P_l) chosen light
$\Phi_p = \frac{\Phi}{n}$, $p(x_p) = \frac{1}{A}$, $p(\omega_p | x_p) = \frac{\cos\theta_p}{\pi}$
Init diffuse → $(\omega'_p, pdf) = $ sample $BSDF(-\omega_p)$, $\Phi'_p = \Phi_p \cdot \cos\theta' \cdot BSDF(-\omega_p, \omega')$
RR → $1 - \min(1, \frac{\Phi'}{\Phi})$ $\Phi' = \Phi$ $L_i(x,\omega) = \sum_{p=1}^{k} f_r(\omega, \omega_{p,i}, \gamma)\frac{\Phi_p}{\pi r^2}$ (pdf, weight)

Kernel: unif, cone, gauss
$\omega_p = h(1 - \frac{\|x_p - q\|}{r})$, $h = \frac{3}{\pi r^2}$
$\omega_p \propto e^{-\frac{1}{2}\frac{\|x_p-q\|^2}{\sigma^2}}$

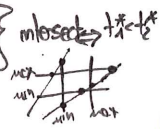
Progressive PM
shrink r_i progressively
$r_{i+1}^2 = \frac{i+\alpha}{i+1} r_i^2$, $\alpha = \frac{2}{3}$ → min MSE
Photon Relax → repulsion force



10. Acceleration DS

$O(N_x N_y N_z)$
Spatial Subdivision = Regular / Adaptive spatial subdivision
• octree • kD tree • BSP
$c = T_t + P_e N_e + P_r N_r$ $\frac{A(Ve)}{A(V)}$
surface area heuristic
- spatial median = primitive median
- SAH

Bounding Volumes: Sphere, AABB, OBB, k-DOP, Chull... → simplex / fit
→ recursive tests
AABB
$t_{x1} = \frac{x_{min} - o_x}{d_x}$, $t_{x2} = \frac{x_{max} - o_x}{d_x}$, t_{y1}, t_{z1} → max t_n^*, t_{y2}, t_{z2} → min t_f^* } intersect ⇔ $t_n^* < t_f^*$



11. Participating Media

$L_i(x,\omega_i) \neq L_o(r(x,\omega_i), -\omega_i)$

Coefficients
- Absorption
- Out scattering
- In Scattering
- Emission

$dL = -\sigma_a L\, dz$
$dL = -\sigma_s L\, dz$
$dL = \sigma_s L_s\, dz$
$dL = \sigma_a L_e\, dz$
$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \left(\frac{1}{\omega}\right)$ $dL = (-\sigma_t L + \sigma_a L_e + \sigma_s L_s)\, dz$
$\sigma_t = \sigma_a + \sigma_s$
$L_z = L_0 e^{-\sigma_t z}$ homog
$q = x + z\omega$, ω: y, x

Transmittance $T_r(x,q) = \begin{cases} e^{-\sigma_t \|y-x\|} & \text{homog} \\ e^{-\int_0^{\|y-x\|} \sigma_t(t)\,dt} & \text{inhomog} \end{cases}$ (multiplicative)

Volume rendering eq

$$L(x,\omega) = \underbrace{T_r(x,x_z)L(x_z,\omega)}_{\text{transmittance}} + \underbrace{\int_0^z T_r(x,x_t)\sigma_a(t)L_e(x_t,\omega)\,dt}_{\text{emission}} + \underbrace{\int_0^z T_r(x,x_t)\sigma_s(t)\int_{S^2} f_p(x_t,\omega',\omega)L_i(x_t,\omega')\,d\omega'\,dt}_{\text{in-scattering}}$$

$L_s(x_t,\omega)$

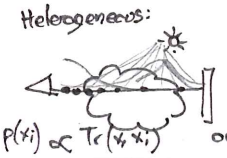
Phase function: distrib of scattered light
$\int_{S^2} f_p(\cdot)\,d\omega = 1$ → isotropic $= \frac{1}{4\pi}$ → anisotropy
$g = \int_{S^2} f_p(\cdot)(-\omega\cdot\omega')$ → how much per angle
$g = 0$ isotr, $g > 0$ forward (skew), $g < 0$ backw (sucks)
Henyey-Greenstein PF $f_p^{HG} = \ldots$

Analytical sol: assume L_i is only direct
- single scattering: $L_i(x,\omega) = T_r(x,p)L_e(p,-\omega)$, $p = r(x,\omega)$
- multiple scattering: in-scattering: $\frac{\Phi}{4\pi}\frac{1}{4\pi}\sigma_s \int_0^z e^{-\sigma_t(\|x-x_t\| + \|x_t - x_{cl}\|)}\,dt$

emitted rad (point light), pf, homog medium

Numerical sol
Ray marching (Riemann sum)
discretize with MC
$T(x,x_i) = T(x,x_{i-1})e^{-\sigma_t(x_i)\Delta t}$
$L_s(x,\omega) \approx \frac{1}{n}\sum_{j=1}^{n} \frac{f_p(x,\omega,\omega_j)L_i(x,\omega_j)}{p(\omega_j)}$

Heterogeneous:

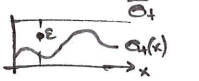


$p(x_i) \propto T_r(x,x_i)$ or $\propto L_i(x_i,\omega)$

Volumetric PT 1) sample dist to next interaction 2) scatter (f_p) or reflect (f_r)

dist: thick medium (milk) → short
$p(t) = \sigma_t e^{-\sigma_t t}$ → $P^{-1}(\varepsilon) = -\frac{\ln(1-\varepsilon)}{\sigma_t}$ $p(x_{t+1}|x_t,\omega) \propto T_r(x_t,x_{t+1}) \Leftrightarrow p(t) \propto T_r(t)$

Woodcock tracking → PT in heterog.: add fictitious volume (make it light → avoid rejection)



$\varepsilon > \frac{\sigma_t(t)}{\bar{\sigma}_t}$ reject, $\varepsilon <$ accept

Volumetric PM 1) trace photons 2) trace cam rays
$L_s \approx \sum_{i=1}^{k} f_p(x_i,\omega_j,\omega)\frac{\Phi_j}{\frac{4}{3}\pi r^3(x)}$ save photons, save none → multiple estimate

Beam Estimate → use cylinder of fixed radius
$L(x,\omega) \approx \cdots + \sum_{i=1}^{N} T_r(x,x_i)\sigma_s(x_i) f_p(x_i,\omega_i,\omega)\frac{\Phi_i}{\pi r^2}$ N beams?
Prefer flexible radius → from r to r_i
Photon Beam → instead of point use ray

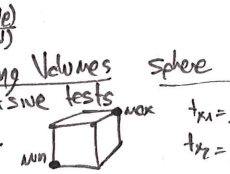
12. Subsurface Scattering

BSSRDF → SS = scattering surface

$S(x_i,\omega_i,x_o,\omega_o) = \frac{dL_o(x_o,\omega_o)}{L_i(x_i,\omega_i)\cos\theta_i\,d\omega}$

$$L_o(x,\omega_o) = \int_A \int_{H^2} S(x_i,\omega_i,x_o,\omega_o) L_i(x_i,\omega_i)\cos\theta\,d\omega_i\,dA(x_i)$$

Diffusion approx → equilibrium distrib in highly scattering optically thick isotropic & semi-infinite
Principle of similarity: simulate anisotropy with iso. $z=0$
$\sigma_s' = (1-g)\sigma_s$, $\sigma_t' = \sigma_a + \sigma_s'$
mean free path $t = 1/\sigma_t'$, $\sigma_t' = \frac{1}{t}$
fluence $\Psi(p_i,p) = \frac{\Phi}{4\pi D}\frac{e^{-\sigma_{tr}\|p_i-p\|}}{\|p_i-p\|} = \int_{S^2} L_i(p_i,\omega_i)\,d\omega_i$
diffusion coeff $D = \frac{1}{3\sigma_t'}$



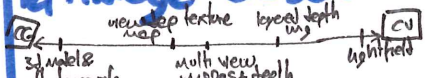
$\Psi(x) = \frac{\Phi}{4\pi D}\left(\frac{e^{-\sigma_{tr}dp}}{dp} - \frac{e^{-\sigma_{tr}dn}}{dn}\right)$
radial symmetric dipole diffusion profile $R_d(x) = \frac{\alpha'}{4\pi}\left(\frac{z_p(1+\sigma_{tr}dp)e^{-\sigma_{tr}dp}}{dp^3} + \frac{z_n(1+\sigma_{tr}dn)e^{-\sigma_{tr}dn}}{dn^3}\right)$
$z_p = \frac{1}{\sigma_t'}$ (mean free path dist) $z_n = z_p(1 + \frac{4}{3}A)$
SS Alg 1) sample irradiance points on surface 2) evaluate diffusion dipole

13. Denoising

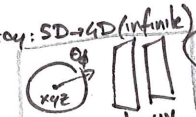
Non-Local Means Filter: noisy $I \to$ estimate $V(i)\ \forall i \in I$
$V(i) = \sum_{j \in I} \omega(i,j)V(j)$ $\omega_{ij} \in [0,1]$, $\sum \omega_{ij} = 1$ similarity
similarity depends on $V(N(i))$, $V(N(j))$ (N = neighborhood)
$\omega_{ij} \propto e^{-\|V(N(i)) - V(N(j))\|^2}$

Prior	Posterior
⊕ good for low spp	⊕ good for high spp
⊖ limited	⊕ general
⊖ intrusive: need info from renderer	⊕ needs few samples (esp variance)

14. Image Based Rendering



Light Field
plenoptic fc: $P(\theta,\phi,x,y,z,\lambda,t)$ 7D time
(dir, pos/origin, spectrum/color)
render any image: pick (x,y,z) cam → $P_{xyz}(\theta,\phi,x,y,z)$ 5D
ray: 5D → 4D (infinite)



Lumigraph 4D plenoptic fc
Virtual image: fix st, move uv
camera array: st discrete, uv continuous → field radiance (dual: surface radiance: fix uv, move st)
Light field rendering: shoot ray ↔ find st, uv, interpolate close val → quadrilinear, 16 values
3D light fields: 3) slice of lumigraph ≡ concentric mosaic
can reconstruct all points inside

⊕ WYSIWYG complex scene/materials (some cost)
⊖ WYSIAYG static scen